

Revisit of Weinberg's sum rules

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Abstract

By applying the Ward identity found by Weinberg two new relations of the amplitude of $a_1 \rightarrow \rho\pi$ with other physical quantities have been found.

One of the most important features revealed from quantum chromodynamics (QCD) is the chiral symmetry in the physics of ordinary hadrons. However, chiral symmetry has been extensively applied to study hadron physics before *QCD* created. Weinberg's sum rules of ρ and a_1 mesons[1] are among those pioneer works. In ref.[1] by applying $SU(2)_L \times SU(2)_R$ and current algebra two Ward identities of vector and axial-vector currents have been obtained and two sum rules have been found. However, in order to obtain the second sum rule(eq.(4) of ref.(1)) one additional assumption has been made[1]. Using VMD and combining Weinberg's sum rules with KSFR sum rule[2], the relationship between the masses of ρ and a_1 mesons $m_{a_1}^2 = 2m_\rho^2$ has been revealed. It predicts that $m_a = 1.09\text{GeV}$. It is well known that a_1 meson has a long history and the present mass of a_1 meson is 1.26 GeV [3]. In this letter we try to see what we can learn if we only apply chiral symmetry, VMD, and current algebra to study the physics of ρ and a_1 mesons. We follow the Ward identity found by Weinberg[1]. From the formalism of ref.[1] it can be seen that the Weinberg's first sum rule is the natural result of chiral symmetry, and current algebra. Under VMD, the first sum rule

$$\int_0^\infty \{\rho_V(\mu^2) - \rho_A(\mu^2)\} \mu^{-2} d\mu^2 = \frac{F_\pi^2}{4} \quad (1)$$

leads to

$$\frac{g_\rho^2}{m_\rho^2} - \frac{g_A^2}{m_A^2} = \frac{F_\pi^2}{4} \quad (2)$$

where

$$\langle 0 | V_\mu^a | \rho_b^\lambda \rangle = \epsilon_\mu^\lambda \delta_{ab} g_\rho,$$

$$\langle 0 | A_\mu^a | a_b^\lambda \rangle = \epsilon_\mu^\lambda \delta_{ab} g_A,$$

and F_π is pion decay constant, $F_\pi = 186\text{MeV}$. This sum rule(2) can be tested. According to VMD, g_ρ is the coupling constant between ρ and γ and it has been determined $g_\rho = 0.12\text{GeV}^2$ from $\Gamma(\rho \rightarrow l^+ l^-)$. Using $m_a = 1.26\text{GeV}$ the first sum rule predicts that

$$g_A = 0.16\text{GeV}^2. \quad (3)$$

On the other hand, g_A can be determined from $\Gamma(\tau \rightarrow a_1 \nu)$. By using VMD, we have

$$\Gamma(\tau \rightarrow a_1 \nu) = \frac{G^2}{8\pi} \cos^2 \theta_c g_A^2 \frac{m_\tau^3}{m_a^2} \left(1 - \frac{m_a^2}{m_\tau^2}\right)^2 \left(1 + 2 \frac{m_a^2}{m_\tau^2}\right). \quad (4)$$

The experimental data of the decay rate is [3] $2.14 \times 10^{-13} (1 \pm 0.32) GeV$. g_A is determined to be $0.16 (1 \pm 0.16) GeV^2$. This value is in good agreement with theoretical prediction (3).

By applying the VMD to the second sum rule of ref.[1]

$$\int_0^\infty \{\rho_V(\mu^2) - \rho_A(\mu^2)\} d\mu^2 = 0 \quad (5)$$

the relationship

$$g_A = g_\rho$$

has been found. Combining eq.(5) with KSFR sum rule [2] the mass relation of ρ and a_1 mesons

$$m_a^2 = 2m_\rho^2 \quad (6)$$

has been established. As mentioned above, the relation (6) is not in good agreement with data. Therefore, it is needed to reexamine the second sum rule. From ref.[1] it can be seen that for the second sum rule (5) besides chiral symmetry and current algebra, an extra assumption has been used. It is not our attempt to comment on this assumption. What we want to study is that if we do not use this assumption and insist on chiral symmetry, VMD, and current algebra, what else can be obtained besides Weinberg's first sum rule (1)? In ref.[1] the following equation (eq.(11) in ref.[1]) has been established by using $SU(2)_L \times SU(2)_R$ chiral symmetry and current algebra

$$\frac{1}{2} q_\mu M^{\mu\nu\lambda}(q, p) = \Delta_V^{\nu\lambda}(q + p) - \Delta_A^{\nu\lambda}(p), \quad (7)$$

where

$$-i\epsilon_{abc} M^{\mu\nu\lambda} = \int d^4x d^4y \langle 0 | T \{ A_a^\mu(x) A_b^\nu(y) V_c^\lambda(0) \} | 0 \rangle \exp\{-iqx - ipy\}, \quad (8)$$

$$\delta_{ab} \Delta_V^{\nu\lambda}(p) = i \int d^4y e^{-ipy} \langle 0 | T \{ V_a^\nu(y) V_b^\lambda(0) \} | 0 \rangle. \quad (9)$$

The first sum rule has been derived from this equation[1]. In this letter the study starts from this equation(7). Following ref.[1] setting $q_\mu = 0$ in eq.(7), on the left hand of the equation(7) only pion poles survive. In this limit the equation (7)now reads

$$\begin{aligned}
& -i\epsilon_{abc}q_\mu M^{\mu\nu\lambda}|_{q_\mu \rightarrow 0} = \\
& \frac{F_\pi}{2k_0} \int d^4y e^{-ipy} k \cdot q \left\{ \frac{\theta(y_0)}{q_0 + k_0 - i\varepsilon} e^{-i(q_0+k_0)y_0} \langle \pi_a(\vec{k} = -\vec{q}, k_0) | A_b^\nu(y) V_c^\lambda(0) | 0 \rangle \right. \\
& + \frac{\theta(-y_0)}{q_0 + k_0 - i\varepsilon} \langle \pi_a(\vec{k} = -\vec{q}, k_0) | V_c^\lambda(0) | A_b^\nu(y) | 0 \rangle \\
& - \frac{\theta(y_0)}{k_0 - q_0 - i\varepsilon} \langle 0 | A_b^\nu(y) V_c^\lambda(0) | \pi_a(\vec{k} = \vec{q}, k_0) \rangle \\
& \left. - \frac{\theta(-y_0)}{k_0 - q_0 - i\varepsilon} e^{i(k_0-q_0)y_0} \langle 0 | V_c^\lambda(0) A_b^\nu(y) | \pi_a(\vec{k} = \vec{q}, k_0) \rangle \right\}. \tag{10}
\end{aligned}$$

In eq.(10) the pion is on mass shell and in chiral limit $k_0 = |\vec{q}|$. When $\vec{k} = -\vec{q}$ there is

$$\frac{k \cdot q}{2k_0} \frac{1}{q_0 + k_0 - i\varepsilon} = \frac{1}{2}, \tag{11}$$

and when $\vec{k} = \vec{q}$ there is

$$\frac{k \cdot q}{2k_0} \frac{1}{k_0 - q_0 - i\varepsilon} = -\frac{1}{2}. \tag{12}$$

In the limit of $q_\mu \rightarrow 0$, the energy of the pion states in eq.(10) is zero. Considering this fact and substituting the two equations(11,12) into eq.(10), the eq.(7) becomes

$$F_\pi \int d^4y e^{-ipy} \langle \pi_a | T \{ A_b^\nu(y) V_c^\lambda(0) \} | 0 \rangle = -\epsilon_{abc} \{ \Delta_V^{\nu\lambda}(p) - \Delta_A^{\nu\lambda}(p) \}. \tag{13}$$

Comparing with the eq.(18) of ref.[1], there is an additional factor of two in eq.(13). This new factor does not affect the second sum rule(5) of ref.[1] under its assumption. However, this factor is important for the study of this letter. In eq.(13) $\Delta_V^{\nu\lambda}(p)$ has ρ meson pole and $\Delta_A^{\nu\lambda}(p)$ has both pion pole and a_1 meson pole. The left hand side of the equation(13) should have these three poles. Therefore, multiplying both sides of the equation by $p^2 - m^2$ and letting $q^2 \rightarrow m^2$ the corresponding pole term can be picked out from the equation,

where m^2 is pion mass(in chiral limit pion mass is zero) , ρ meson mass, and a_1 meson mass respectively. The poles of the left hand side of the eq.(13) can be obtained in following way

$$\begin{aligned}
F_\pi \int d^4y e^{-ipy} < \pi_a | T \{ A_b^\nu(y) V_c^\lambda(0) \} | 0 > = \\
\{ \theta(y_0) e^{-iky} i F_\pi k_\nu < \pi_a \pi_b(k) | V_c^\lambda(0) | 0 > + \theta(y_0) e^{iky} (-) i F_\pi k_\nu < \pi_a | V_c^\lambda(0) | \pi_a > \\
+ \theta(y_0) e^{-iky} g_\rho \varepsilon_\lambda^{\sigma*}(k) < \pi_a | A_b^\nu(0) | \rho_c^\sigma(k) > + \theta(y_0) e^{iky} g_\rho \varepsilon_\lambda^\sigma(k) < \pi_a \rho_c^\sigma(k) | A_b^\nu(0) | 0 > \\
+ \theta(y_0) e^{-iky} g_A \varepsilon_\nu^\sigma(k) < \pi_a a_b^\sigma(k) | V_c^\lambda(0) | 0 > + \theta(y_0) e^{iky} g_A \varepsilon_\nu^{\sigma*}(k) < \pi_a | V_c^\lambda(0) | a_b^\sigma(k) > \}
\end{aligned} \quad (14)$$

After using VMD, the following formulas are obtained

$$\begin{aligned}
< \pi_a | V_c^\lambda(0) | \pi_b(k) > &= i \epsilon_{abc} g_\rho f_{\rho\pi\pi} \frac{k^\lambda}{m_{m_\rho^2}}, \\
< \pi_a \pi_b(k) | V_c^\lambda(0) | 0 > &= -i \epsilon_{abc} g_\rho f_{\rho\pi\pi} \frac{k^\lambda}{m_{m_\rho^2}},
\end{aligned} \quad (15)$$

where $f_{\rho\pi\pi}$ is the coupling constant of the decay of $\rho \rightarrow 2\pi$ in the limit of that the energy of one pion is zero. Using VMD, the other four matrix elements in eq.(14) are related to the decay of $a_1 \rightarrow \rho\pi$. The vertex of this decay has been written as

$$\{ A g_{\mu\nu} + B p_{\pi\mu} p_{\pi\nu} \} \epsilon_{abc} a_\mu^a \rho_\nu^b \pi_c, \quad (16)$$

where A and B are functions of momenta of a_1 , ρ , and π . In eq.(14), the energy of the state $|\pi_a\rangle$ is zero, therefore only the amplitude A contributes to these matrix elements of eq.(14) and the amplitude A is in the limit of $p_\pi = 0$. Using eq.(16) and VMD we obtain

$$\begin{aligned}
< \pi_a | A_b^\nu(0) | \rho_c^\sigma(k) > &= -\epsilon_{abc} g_A A(m_\rho^2) \varepsilon^\sigma(k)^\nu \frac{1}{m_\rho^2 - m_a^2}, \\
< \pi_a \rho_c^\sigma(k) | A_b^\nu(0) | 0 > &= -\epsilon_{abc} g_A A(m_\rho^2) \varepsilon^{\sigma*}(k)^\nu \frac{1}{m_\rho^2 - m_a^2}.
\end{aligned} \quad (17)$$

In eq.(17), due to $k^2 = m_\rho^2$, we have $A(k^2) = A(m_\rho^2)$. In the same way, two other matrix elements of eq.(14) can be written as

$$< \pi_a | V_c^\lambda(0) | a_b^\sigma(k) > = -\epsilon_{abc} g_\rho A(m_a^2) \frac{1}{m_a^2 - m_\rho^2} \varepsilon^\sigma(k)^\lambda,$$

$$\langle \pi_a a_b^\sigma(k) | V_c^\lambda(0) | 0 \rangle = -\epsilon_{abc} g_\rho A(m_a^2) \frac{1}{m_a^2 - m_\rho^2} \epsilon^{\sigma*}(k)^\lambda. \quad (18)$$

It is necessary to point out that due to the limit of $p_\pi = 0$ in eq.(13), the amplitude A is function of k^2 where k is either the momentum of ρ or a_1 . It has been found that in eq.(17) $k^2 = m_\rho^2$ and $k^2 = m_a^2$ in eq.(18). Substituting the six matrix elements(15,17,18) into eq.(14), the pole terms on the left hand side of the eq.(13) are obtained

$$\begin{aligned} F_\pi \int d^4 y e^{-ipy} \langle \pi_a | T \{ A_b^\nu(y) V_c^\lambda(0) \} | 0 \rangle|_{poles} &= i g_\rho f_{\rho\pi\pi} \frac{F_\pi^2}{m_\rho^2} \frac{p^\nu p^\lambda}{p^2} \\ &- i \epsilon_{abc} F_\pi g_\rho g_A \frac{1}{m_\rho^2 - m_a^2} \frac{1}{p^2 - m_\rho^2} (-g^{\lambda\nu} + \frac{p^\lambda p^\nu}{m_\rho^2}) \\ &- i \epsilon_{abc} F_\pi g_\rho g_A A(m_a^2) \frac{1}{m_a^2 - m_\rho^2} \frac{1}{p^2 - m_a^2} (-g^{\nu\lambda} + \frac{p^\nu p^\lambda}{m_a^2}). \end{aligned} \quad (19)$$

By applying the VMD to the right hand side of the equation(13), the pole terms are obtained

$$i F_\pi^2 \frac{p^\nu p^\lambda}{p^2} + i g_\rho^2 \frac{1}{p^2 - m_\rho^2} (-g^{\nu\lambda} + \frac{p^\nu p^\lambda}{p^2}) i g_A^2 \frac{1}{p^2 - m_a^2} (-g^{\nu\lambda} + \frac{p^\nu p^\lambda}{m_a^2}). \quad (20)$$

Multiply both sides of eq.(13) by p^2 and set $p^2 = 0$ the pion pole can be picked out and the following relation can be found

$$g_\rho f_{\rho\pi\pi} = m_\rho^2. \quad (21)$$

Multiply eq.(13) by $p^2 - m_\rho^2$ and set $p^2 = m_\rho^2$ and the ρ pole can be picked out and the second relation is obtained

$$g_A F_\pi A(m_\rho^2) = -g_\rho (m_a^2 - m_\rho^2). \quad (22)$$

Multiply eq.(13) by $p^2 - m_a^2$ and set $p^2 = m_a^2$ and the a_1 pole can be picked out and the third relation is found

$$g_\rho F_\pi A(m_a^2) = -g_A (m_a^2 - m_\rho^2). \quad (23)$$

Let's discuss these three relations. The first relation is well known from VMD[4]. In eqs.(22,23), if we assume that $A(k^2)$, where k is either the momentum of ρ meson or the

momentum of a_1 meson, is independent of k^2 , i.e.

$$A(m_\rho^2) = A(m_a^2), \quad (24)$$

then from eqs.(22,23) it can be obtained that

$$g_\rho = g_A$$

which is just the consequence of the second sum rule of ref.[1]. However, based on chiral symmetry, VMD, and current algebra only, we can not reach the conclusion(24). On the other hand, from the discussion of Weinberg's first sum rule above and the current value of m_a , the relation (24) does not have good support theoretically and experimentally. Therefore, in general, $A(k^2)$ depends on k^2 . From eqs.(22,23), we determine that

$$A(m_\rho^2) = -8.14 GeV, \quad A(m_a^2) = -14.05 GeV.$$

Therefore, $A(k^2)$ strongly depends on k^2 . In order to see the deviation of these values from physical ones we can use the experimental data of the ratio of d-wave to s wave[5] $d/s = -0.11 \pm 0.02$ to determine the B in the amplitude (16),

$$B = -1.1A.$$

Using the values of A and B, the width of $a_1 \rightarrow \rho\pi$ calculated is higher than experimental value[3] by one order of magnitude. Of course, the A's in eqs.(22,23) are determined in an unphysical limit of $p_\pi = 0$. Unlike the case of $\rho \rightarrow \pi\pi$ that $f_{\rho\pi\pi}(p_\pi = 0)$ determined in eq.(21) and KSFR sum rule is very close to physical value, the A's determined by eqs.(22,23) in the limit of $p_\pi = 0$ are far away from the physical value of the amplitude. In ref.[6] a chiral theory of mesons including pseudoscalar, vector and axial-vector mesons has been studied and it can be considered as a realization of chiral symmetry, VMD, and current algebra. In this theory Weinberg's first sum rule and relation (21) are satisfied. It can also be seen that

the amplitude A found in this theory satisfies the relations (22,23) in the limit of $p_\pi = 0$. Explanations of why $f_{\rho\pi\pi}$ in the limit of $p_\pi = 0$ is very close to the physical value and the amplitude A in the limit of $p_\pi = 0$ is not, can be found and a new mass relation of ρ and a_1 mesons has been presented.

To conclude, relations(22,23) do not lead to $m_a^2 = 2m_\rho^2$, therefore, from chiral symmetry, VMD, and current algebra alone this mass relation can not be obtained. In the limit of $p_\pi = 0$, the amplitude of $a_1 \rightarrow \rho\pi$ must satisfy the relations(22,23).

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